Peak Deviation from Prediction in Atomic Clocks*

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***Abstract*— Results from a preliminary study of two GPS clocks tested on the ground show that the distribution of peak deviations from prediction cannot be assumed to be Gaussian. The standards are identical to models on board GPS block IIR satellites. The data were taken at the Naval Research Laboratory in the U.S. during life tests. We propose here two statistics that could be used to characterize peak deviation from prediction, preliminarily named PDIS and PTIE.**

I. Introduction

Atomic clocks are most often characterized in terms of a deviation, a root-mean-squared (RMS) second moment measure. Some applications require statistics of peak error performance, a first moment measure. For example telecommunications systems specify maximum time interval error (MTIE) limits for clocks. MTIE specifies the performance of a slave clock that is locked to a master. Free-running clocks with a prediction of their expected performance represent a different problem. This is the case with clocks on Global Positioning System (GPS) satellites, where the clocks are generally free-running, but the system depends on their performance matching the uploaded prediction. We propose here a method for specifying the deviation from prediction. In applying this to two Rb bulb atomic standards, we find that one cannot assume that distributions of the deviations from prediction are Gaussian.

We first need a method to predict the time offset of a clock from a reference, given measurements $x(t)$, $0< t < T$. Suppose we want a prediction, $x_f(T+\tau)$, of what the value will be at time $T+\tau$, an amount τ after the last measurement. We then measure the actual value $x(T+\tau)$, and compute the residual difference. We may repeat this to obtain a set of residuals for a given prediction interval τ. If this set significantly differs from a Gaussian distribution, then we cannot use Gaussian statistics to characterize the performance of clocks over a prediction interval. In particular, one cannot use the familiar Gaussian Normal distribution to determine the probability of exceeding a threshold offset from the number of standard deviations. The relationship between the variance or deviation of a clock and the probability of missing a prediction by a certain amount depends strongly on the distribution of the residuals.

II. Statistical Method

A. Optimal and Near-Optimal Prediction

We consider a clock model with a combination of deterministic linear frequency drift and a sum of the stochastic noise types of white frequency modulation (FM), flicker FM, and random walk FM. Thus the normalized frequency of the clock behaves as

$$y(t) = y_0 + D \bullet t + \varepsilon(t), \qquad (1)$$

where y_0 is the initial normalized frequency at time t=0, D is the drift rate, and ε is the stochastic term. The drift is first estimated and removed from the data. A near-optimal drift estimator in the presence of these noise types is due to Greenhall [1]. If $x(t)$ is the time departure of the clock, so $y(t) = dx(t)/dt$, define $w(t)$ as

$$w(t) = \int_0^t x(\lambda)\, d\lambda. \qquad (2)$$

Then, from Greenhall, if the data length is T, the estimate of drift is the following linear combination of four points from w,

$$D = \frac{50}{3T^3} \bullet \left[4w(T) - 4w(0) - 5w\left(\frac{9T}{10}\right) + 5w\left(\frac{T}{10}\right) \right]. \qquad (3)$$

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Once the drift is removed, the spectrum $S_y(f)$ of y(t) satisfies [2]

$$S_y(f) = h_0 f^0 + h_{-1} f^{-1} + h_{-2} f^{-2}, \qquad (4)$$

where h_i is the coefficient of the power-law noise process with exponent i. An optimal or near-optimal filter for predicting $x(T+\tau)$, given $x(t)$, $0<t<T$, employs an exponential filter on $y(t)$ [3]. In practice we have discrete-time phase data $x_n = x(n\tau_0)$, for $n = 1, \dots, N$. Then the discrete version of $y(t)$ is $y_n = (x_n - x_{n-1})/\tau_0$, for $n = 2, \dots, N$.

We need a normalized frequency at N, optimal or near-optimal for prediction. The optimal predictor for white FM is the mean frequency; for random-walk FM, the last frequency; for flicker FM, a filter with some a fading memory of the last few frequencies. An exponential filter producing an estimate y_{fn} of y_n, in which the white FM has been averaged down to the low-frequency dominated noise types is

$$y_{fn} = \frac{y_n + K(y_{fn-1} + D)}{1+K}, \qquad (5)$$

where D is the drift in appropriate units, and K is the half-life time of the exponential filter. In the case of pure random-walk FM, $S_y(f) = h_{-2}/f^2$, the optimal predictor is the last normalized frequency, $y_{fn} = y_n$. In this case we have $K = 0$. In the case of pure white FM, $S_y(f) = h_0$, the optimal predictor is the mean. y_{fn} approaches the mean as $K \to \infty$. In the general case white FM dominates in the short term, and flicker or random-walk FM dominate as we increase integration time. We choose K as the integration time where this transition happens. This can be determined by computing the Allan deviation of the drift removed data, or the Hadamard deviation of the data, even with linear frequency drift. K is chosen as the integration time τ where the deviation no longer drops with increasing τ, sometimes called the "knee" of the deviation.

We wanted a prediction of $x(T+\tau)$. Let us view this in terms of discrete measurements. Then we want a prediction of x_{N+k}, where $\tau = k\tau_0$. We use

$$x_{fN+k} = x_N + k y_{fN} + D \bullet \frac{k^2}{2}. \qquad (6)$$

B. *Toward a Peak Deviation Standard*

If atomic clocks are to be required to stay within a specified range of prediction with a specified probability, it is necessary to define a statistic that will measure this. We discuss two candidates for such statistics. The method described above involves using data taken at points numbered $1, \dots, N$ to predict a value later at $N+k$. In practice, rather than predicting a future measurement and waiting for that, we may use the same formalism with a large data set x_n, $n = 1, \dots, M$, and compute residuals

$$x_{fN+k} - x_{N+k}, \qquad (7)$$

for all $N+k < M$ of interest. For a given prediction interval k, we may plot a histogram of actual residuals and compare it to a best fit of a Gaussian distribution. Thus, for a range of prediction intervals, we obtain a set of distribution plots giving the frequency at which this clock deviates from prediction by what values. We call this set of plots PDIS, a set of peak distribution functions. This would be most useful in designing a system requiring limits on peak deviation from prediction. Let us call a plot of the maximum absolute value of deviation as a function of prediction interval PTIE, the peak time-interval error, analogous to MTIE. This could be used as a specification that a clock should not exceed during a qualification test.

III. A Preliminary Study of Two Clocks

We studied two GPS Rb cell atomic clocks, R28 and R30, using some data from life-tests on the ground at the Naval Research Lab (NRL). We computed the peak deviation from a near-optimal prediction of the time offset of the clocks from the reference as discussed in Section II. Clock signals from GPS are corrected by predictions that are uploaded. Between uploads, the peak deviation from the prediction gives the worst-case range error due to the clock.

We estimated the linear frequency drift for each clock as a constant over the data interval. This was done by use of a four-point estimator, as in [1]. Figure 1 plots the data for the two clocks with the drift removed. Rb clock R28 appears to have had two visible shifts in drift during the period of the data set. With post-processing, this can be detected and adjusted for. However, in a real-time system it is very difficult to detect a change in drift until some period after the change. Hence we continued the analysis with one constant drift value.

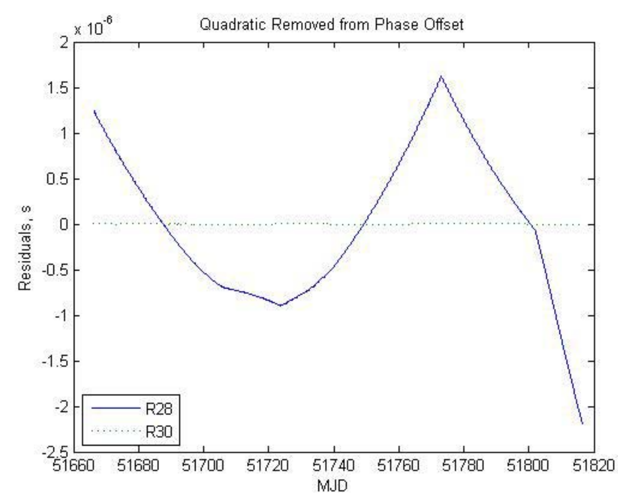

Figure 1 Clock phase with drift removed, for Rb clocks R28 and R30, showing the consistency of the linear frequency drift over the data sets.

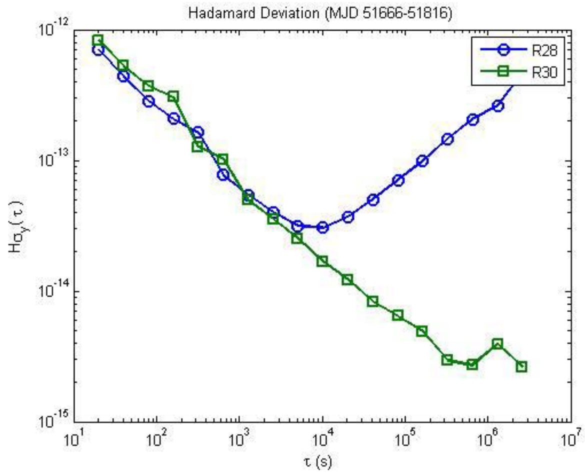

Figure 2 Hadamard deviation from life-test Rb clocks R28 and R30.

Figure 2 shows the Hadamard deviation of the two data sets. The "knee" of the deviation is significantly different between the two clocks. 10^4 s and $4x10^5$ s were used as the half-life averaging times in the exponential filters.

We computed the peak deviation over the following prediction intervals: 15 minutes, 1 hour, 2 hours, 4 hours, 8 hours, and 1 day. The two clocks were Block IIR Rb. clocks, serial numbers R28 and R30. They were chosen as one of the worst, R28, and the best, R30, clocks from the qualification tests. The data sets were 20 s time offsets from the reference H-Maser over Modified Julian Days (MJDs) 51666- 51816. In Figures 3 and 4 we show the filtered normalized frequency on top of the normalized frequency estimates from first differences over 20 s.

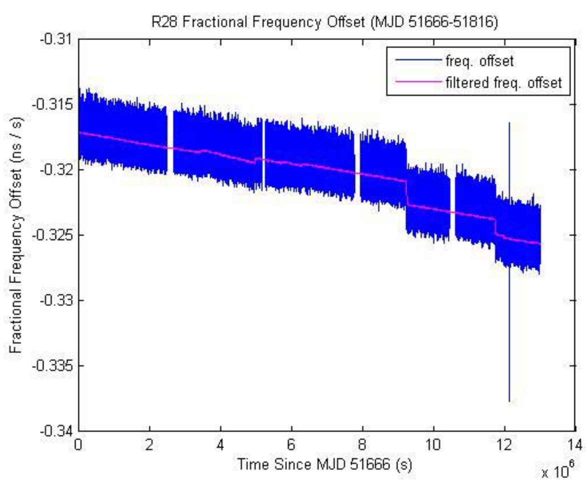

Figure 3 Rb clock R28 normalized frequency offset from first differences of 20 s measurements and from the filter.

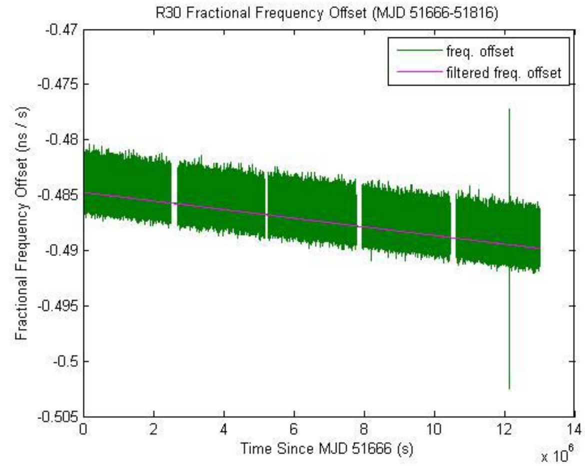

Figure 4 Clock R30 normalized frequency offset from 1st differences of 20 s measurements and from the filter.

For R28, in Figure 3, we see a number of frequency steps. For R30 in Figure 4 we note that the filtered estimate does not go through the mean or center of the first difference normalized frequency offset data. These two effects result in non-Gaussian peak deviation errors, for different reasons. Frequency steps cause peak prediction error in time data, because the predictor cannot incorporate knowledge of an unexpected step change in frequency. Hence we find that the longer the prediction interval the worse the error.

As discussed in Section II, we computed the distribution of peak prediction errors as follows:

1. Estimate drift from the entire data set.
2. Pick a prediction interval, *τ,* among: 15 minutes, 1 hour, 2 hours, 4 hours, 8 hours, 1 day.
3. For each 20 s measurement at time *t*, take the phase offset, *x(t)*, from the reference at that time and use the filtered normalized frequency at that time, $y_f(t)$, and estimated linear frequency drift, *d*, from 1. to predict time forward as a quadratic over the given prediction interval. This is the predicted time.
4. The peak deviation error at $t+\tau$, over the prediction interval τ, is the residual, the difference between the predicted and the measured time offset.
5. We collect the *residuals* for a given prediction interval τ over all of the possible values of *t* in the data set. Plotting these as a histogram gives the distribution function. The set of distribution functions is PDIS.

The plots in Figure 5 show prediction error as a function of date for different prediction intervals for R28. These plots lead to the PDIS distribution functions for R28 in Figure 6. The dashed lines are a best-fit Gaussian distribution to the data. This set of plots is PDIS for R28.

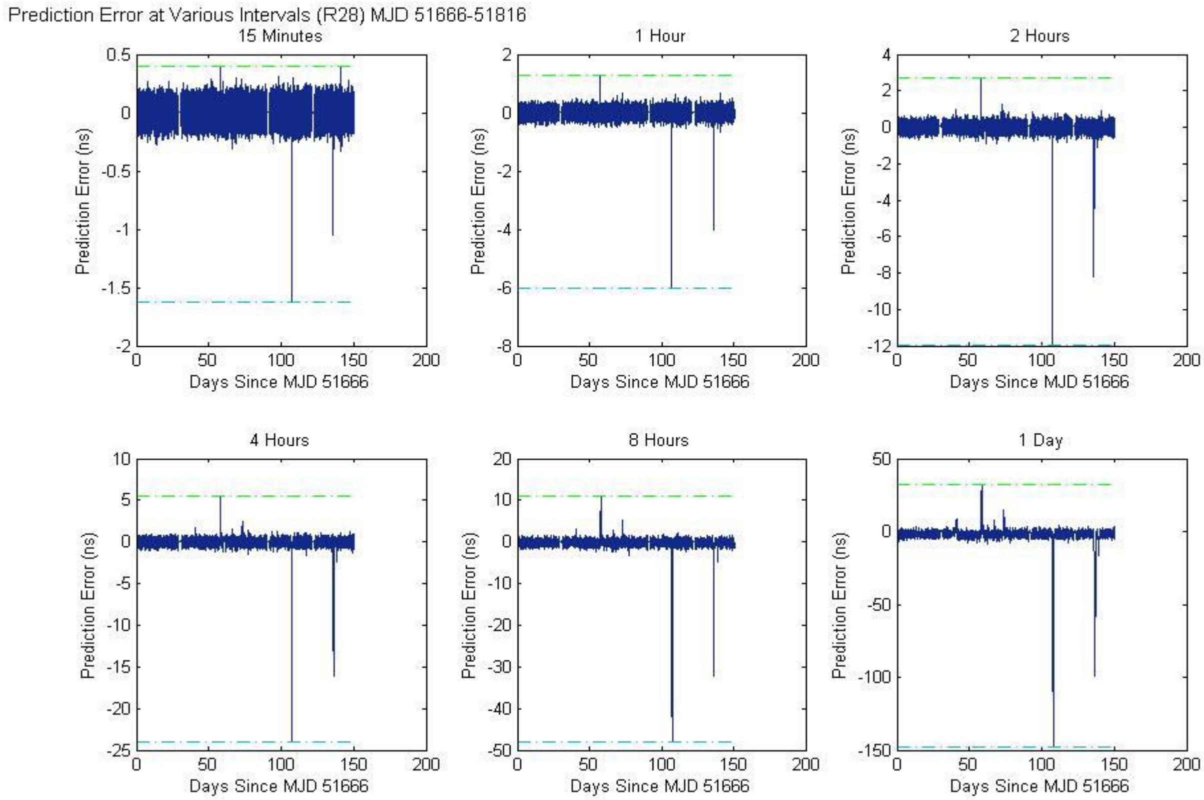

Figure 5 Prediction error for various prediction intervals, showing when the errors occur.

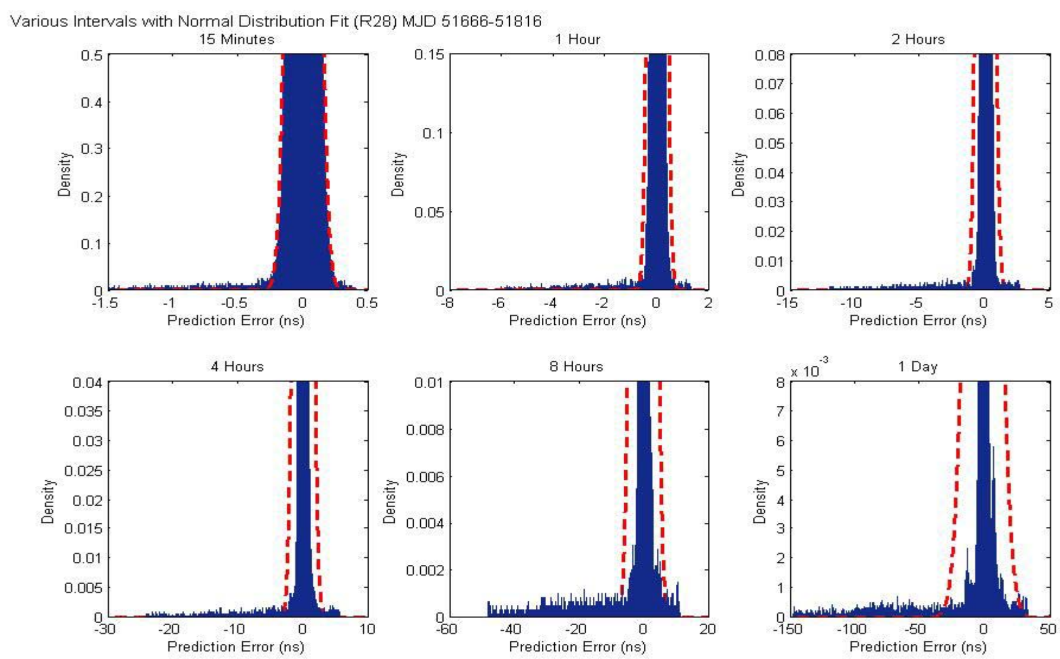

Figure 6 PDIS for clock R28, the distribution of prediction errors for various prediction intervals. The dashed lines show best fits for Gaussian distributions.

For R30 we found an effect, different from that for R28. R30 had no significant frequency steps over the study interval. However, as we noted above, the filtered normalized frequency estimate did not go through what appeared to be the center of the normalized frequency from first differences of measurements. To understand this, consider Figure 7, the distribution of all of the 20 s residuals of first differences of measurements minus the filtered estimates of normalized frequency.

The high peak at around prediction error 0 occurs because of data gaps. Since we interpolated across them, the filter and measurement were aligned. However, note the bimodal nature of the distribution. The clock appears to have two modes separated by almost 3 x 10^{-12} in normalized frequency. This results in the distributions of time deviations having a non-zero mean. Figure 8 presents PDIS for R30. The plots appear approximately Gaussian, but for larger prediction times have increasingly non-zero means.

Figure 9 gives the PTIE values for R28 and R30 based on the PDIS values of Figures 6 and 8. PTIE is the plot of the maximum absolute value of deviation as a function of prediction time.

IV. Conclusions

We conclude that one cannot assume that Gaussian statistics are valid for peak deviation of atomic clocks from prediction. Hence we cannot look at a root-mean-squared measure of clock performance, such as the Allan variance, and use that to predict probabilities of peak time deviations. To determine failure probabilities for integrity it will be necessary to characterize the distribution of peak deviation from prediction errors for the design clocks. It will also be necessary to incorporate peak testing into acceptance tests.

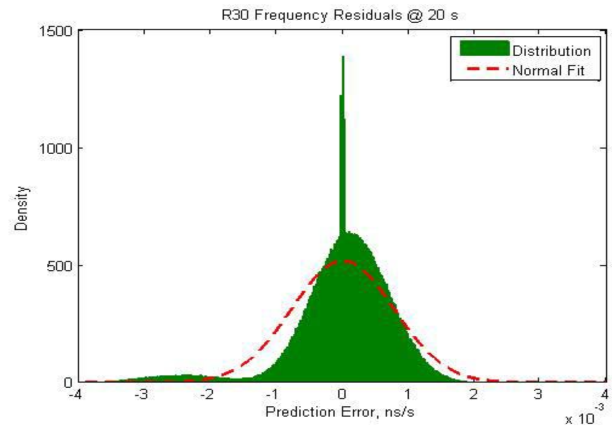

Figure 7 Normalized frequency residuals for clock R30 showing the bimodal behavior of the clock. The dashed line is a best fit for a Gaussian distribution.

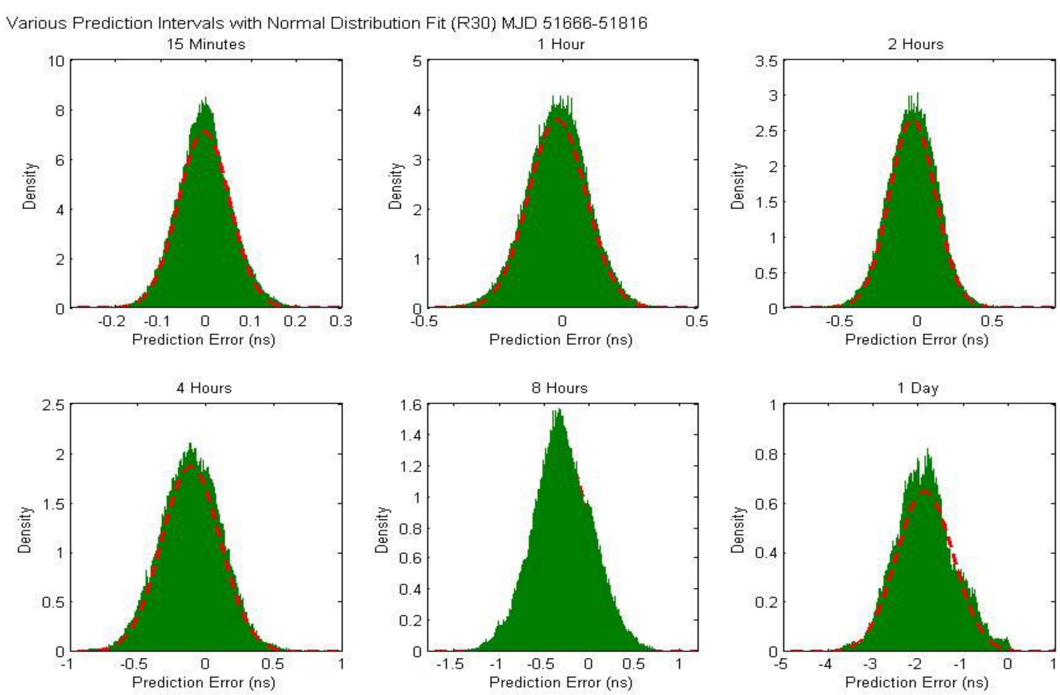

Figure 8 PDIS for clock R30, the distribution of prediction errors for various prediction intervals.

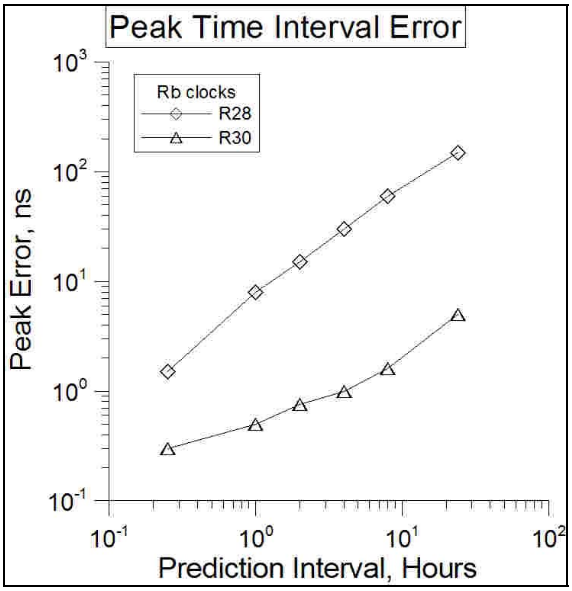

Figure 9 PTIE for clocks R28 and R30 from this study.

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